plates than thin ones. For a side-to-thickness ratio of 5, maximum deflection occurs at 15 deg for graphite/epoxy and about 13 deg for Aramid/rubber. For moderately thick and thin plates, the maximum deflection shifts to about 27 deg for graphite/expoxy and 35 deg for Aramid/rubber.

Finally, to demonstrate the difference between the bimodular and "unimodular" assumptions, comparisons were made among the dimensionless center deflections of simply supported, square, two-layer, angle-ply plates under uniformly distributed load using average, tensile, compressive, and bimodular properties (see Table 3). The results from the two models, namely bimodular and unimodular, differ slightly from each other for graphite/epoxy (up to about 6%). However, for Aramid/rubber, which has a bimodularity ratio of about 300, the difference increases up to 88%.

Concluding Remarks

In this study, a displacement finite element model is developed for bending of thick, rectangular, antisymmetric, angle-ply, fiber-reinforced, bimodular composite plates. By examining the results, it is concluded that the effects of plate aspect ratio, bimodularity ratio, fiber orientation, and shear deformation are considerable on the transverse deflection. In addition, the result shows that the influence of bimodularity ratio on shear deformation is minimal. Furthermore, for materials with a high bimodularity ratio, a unimodular assumption may result in a very large error in the displacement prediction.

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Instability of a Partially Delaminated Surface Layer of an Oscillating Cylinder

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Introduction

THE subject of delamination buckling and growth has been the topic of many investigations over the past few years, primarily in the context of flat layers. Recently, however, several studies pertaining to cylindrical layers have been conducted.[1-7] (An extensive list of references to the field can be found in Refs. 3 and 4.) These studies have been concerned primarily with the buckling[1,2,7] and quasistatic growth[3-6] of thin-film delaminations under various loading and kinematical conditions. Aspects of the dynamics of the phenomena (both buckling and growth[8] and vibration about buckled states[9]) have been considered for specific configurations of flat layers as well. It appears that the case of buckling and growth induced by periodic loading, however, has not been addressed similarly.

Here, we consider the problem of a thin elastic layer adhered to the surface of a cylindrical substrate such that a "thru" delamination exists over a portion of the interface and the surface of the substrate oscillates uniformly. It shall be assumed for simplicity that the gap between the layer and the substrate in the damaged region is sufficiently deep to prevent contact for small deflections of the film. In what follows, the substrate is considered to be extremely stiff relative to the layer, such that its behavior is effectively unaltered by the presence of the layer. Thus, the small amplitude motion of the surface of the substrate is considered as prescribed and uniform.

The shallow arch theory employed as the mathematical model of the layer in the recent studies concerning delamination of cylindrical layers[4-6] shall be adopted here, and the corresponding equations of motion derived. Buckling, or instability, of the layer (i.e., the onset of large radial deflections of the layer measured with respect to the substrate) shall be

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assessed in the context of the boundedness of the solutions to the equations of motion (see, for example, Refs. 10–12), and regions of instabilty in the parameter space of the system shall be obtained by employing the results of an asymptotic analysis.[10]

Problem Formulation

Consider a thin elastic layer adhered to the surface of a rigid but uniformly and periodically contracting and expanding cylindrical substrate, and let a thru delamination exist over the angular interval $[-\phi,\phi]$. Let the prescribed radial deflection of the surface of the substrate be given by $w_o(\tau)$ (positive inward), where τ represents the normalized time as defined later. The equations of motion governing the delaminated segment of layer are derived by applying Hamilton's principle, which in the present context takes the form

$$\bar{\delta}\int_{\tau_1}^{\tau_2}\int_{-\phi}^{\phi}\left(\frac{1}{2}\kappa^2+\frac{1}{2C}N^2-\frac{1}{2}mv^2\right)\mathrm{d}\theta\,\mathrm{d}\tau=0 \tag{1}$$

where $\bar{\delta}$ represents the variational operator, and the normalized curvature change κ, normalized membrane force N, and centerline particle speed v of the layer segment are given in terms of the normalized circumferential and radial deflections of the layer centerline $u(\theta,\tau)$ (positive clockwise) and $w(\theta,\tau)$ (positive inward) as

$$\kappa = w,_{\theta\theta} + w \tag{2a}$$

$$N = -C(u,_\theta - w + \tfrac{1}{2}w,_\theta^2) \tag{2b}$$

$$v^2 = u,_\tau^2 + w,_\tau^2 \tag{2c}$$

In Eqs. (1) and (2), $C=12/h^2$ is the normalized membrane stiffness of the layer, $h \ll 1$ the normalized thickness of the layer, $m = C/4\phi^2$ the normalized mass per unit length of the layer, and $F,_\theta$ denotes partial differentiation with respect to the angular coordinate θ, etc. In the previous expressions, all length scales have been normalized with respect to the radius of the layer in its undeformed configuration, and $\tau = v_o t/2\phi$ is the normalized time where $v_o^2 = E/\rho$, E and ρ correspond to Young's modulus and the mass density of the layer, respectively, and t represents the (dimensional) time. It may be seen from its definition that $\tau = 1$ corresponds to the time required for a circumferential wave to traverse the delaminated segment of layer. Performing the associated operations leads to the pair of coupled nonlinear partial differential equations governing $w(\theta,\tau)$ and $u(\theta,\tau)$ given by

$$w,_{\theta\theta\theta\theta} + (2+N)w,_{\theta\theta} + N,_\theta w,_\theta + w + N + mw,_{\tau\tau} = 0 \tag{3a}$$

$$N,_\theta + mu,_{\tau\tau} = 0 \tag{3b}$$

together with the boundary conditions

$$u\,|_{\theta=\pm\phi} = 0 \tag{4}$$

$$w\,|_{\theta=\pm\phi} = w_o(\tau) \tag{5a}$$

$$w,_\theta|_{\theta=\pm\phi} = 0 \tag{5b}$$

In the ensuing analysis, we shall neglect the circumferential inertial effects in the layer. This is equivalent to neglecting the kinetic energy of circumferential motion of a material particle on the centerline of the layer, compared with the corresponding kinetic energy of radial motion, in Eq. (1). Under such conditions, it may be seen from Eq. (3b) that the resultant membrane force is spatially uniform throughout the delaminated segment of layer. Equations (3a) and (3b) thus become

$$w,_{\theta\theta\theta\theta} + (2+N_o)w,_{\theta\theta} + w + N_o + mw,_{\tau\tau} = 0 \tag{6a}$$

$$N = N_o(\tau) \tag{6b}$$

Upon noting Eq. (2b), the boundary conditions for u, Eq. (4), can be replaced by the integrability condition defined by

$$\int_{-\phi}^{\phi}(w - \tfrac{1}{2}w,_\theta^2)\,\mathrm{d}\theta - 2\phi N_o/C = 0 \tag{7}$$

The problem is thus transformed to one in terms of $w(\theta,\tau)$ and $N_o(\tau)$. We shall be interested in the circumstances under which the solutions to the system defined by Eqs. (5–7) grow without bound (i.e., when the deflections of the layer become large).

Analysis

We shall next assess the stability of the debonded segment of layer for the case of spatially uniform and temporally periodic contractions and expansions of the substrate boundary, where stability shall be defined in terms of the boundedness of the radial deflections $w(\theta,\tau)$. Specifically, the delaminated segment of layer will be said to become unstable when the solutions to Eq. (6a) increase without bound with increasing time (that is, when $w \to \infty$ as $\tau \to \infty$). This shall be done by first expanding $w(\theta,\tau)$ about the spatially uniform deflection $w_o(\tau)$ in terms of the modal functions associated with the corresponding linear problem and will result in a system of coupled ordinary differential equations for the corresponding time-dependent amplitudes of the modes. Conditions will then be determined for these amplitudes to grow without bound.

Let us first assume a solution to the system [Eqs. (6)] in the form

$$w(\theta,\tau) = w_o(\tau) + \sum_i q_i(\tau)\bar{v}_i(\theta) \tag{8a}$$

$$N_o(\tau) = Cw_o(\tau) + \sum_i q_i(\tau)\bar{P}_i \tag{8b}$$

where

$$\bar{v}_i(\theta) = v_i(\theta)/\|v_i\| \tag{9a}$$

$$\bar{P}_i = \bar{P}(\omega_i) = P(\omega_i)/\|v_i\| \tag{9b}$$

$$\|v_i\| = \left[\int_{-\phi}^{\phi} v_i^2(\theta)\,\mathrm{d}\theta\right]^{1/2} \tag{9c}$$

The corresponding (mutually orthogonal) modal functions and associated frequency equations were found by solving the linearized unforced version of Eqs. (5) and (6) assuming a harmonic time dependence, and are presented below.

Symmetric Modes

$$v_i^{(s)}(\theta) = \beta_i \sin\beta_i\phi(\cos\alpha_i\theta - \cos\alpha_i\phi)$$
$$- \alpha_i \sin\alpha_i\phi(\cos\beta_i\theta - \cos\beta_i\phi) \qquad (i = 1,2,\ldots) \tag{10a}$$

$$P_i^{(s)} = P^{(s)}(\omega_i) = (1-\omega_{(i)}^2 m)[\beta_i \sin\beta_i\phi \cos\alpha_i\phi$$
$$-\alpha_i \sin\alpha_i\phi \cos\beta_i\phi] \qquad (i = 1,2,\ldots) \tag{10b}$$

where

$$\alpha_i = \left[1 - \omega_i\sqrt{m}\right]^{1/2} \qquad (i = 1,2,\ldots) \tag{11a}$$

$$\beta_i = \left[1 + \omega_i\sqrt{m}\right]^{1/2} \qquad (i = 1,2,\ldots) \tag{11b}$$

and the frequencies ω_i correspond to the ith root of the characteristic equation defined by

$$F^{(s)}(\omega) \equiv \alpha\beta\,\frac{\phi(1-\omega^2 m + C)}{C(1-\omega^2 m)^2}\,P^{(s)}(\omega)$$
$$- 2\omega\sqrt{m}\,\sin\alpha\phi\,\sin\beta\phi = 0 \tag{12}$$

Antisymmetric Modes

$$v_i^{(a)}(\theta) = \sin\beta_i\phi\,\sin\alpha_i\theta - \sin\alpha_i\phi\,\sin\beta_i\theta \quad (i = 1,2,\dots) \tag{13a}$$

$$P_i^{(a)} = 0 \qquad (i = 1,2,\dots) \tag{13b}$$

where α_i and β_i are defined by Eqs. (11a) and (11b) and ω_i corresponds to the ith root of the characteristic equation defined by

$$F^{(a)}(\omega) \equiv \beta\,\cos\beta\phi\,\sin\alpha\phi - \alpha\,\cos\alpha\phi\,\sin\beta\phi = 0 \tag{14}$$

Next, we substitute Eqs. (8a) and (8b) into Eq. (6a), multiply by $\bar{v}_j(\theta)$, and integrate over $[-\phi,\phi]$. Retaining terms up to $\mathcal{O}(q_i)$ and making use of the fact that the modal functions $\bar{v}_i$ $(i=1,2,\dots)$ are mutually orthonormal, we arrive at the coupled system of ordinary differential equations governing the modal amplitudes $q_i(\tau)$ $(i=1,2,\dots)$. We thus have

$$\ddot{q}_j + \sum_i \left[\omega_{(j)}^2\delta_{ji} + 4\phi^2 w_o(\tau) R_{ji}\right] q_i = \frac{-2\phi}{C}\bar{P}_j \times \left[4\phi^2 w_o(\tau) + \ddot{w}_o(\tau)\right] \qquad (j = 1,2,\dots) \tag{15}$$

where

$$R_{ji} = \int_{-\phi}^{\phi} \bar{v}_j \bar{v}_i''\,\mathrm{d}\theta \tag{16}$$

δ_{ji} represents Kronecker's delta, superposed primes denote total differentiation with respect to θ, and superposed dots denote total differentiation with respect to τ. We note that the right side of Eq. (15) vanishes for antisymmetric modes.

At this point, let us specify the deflection of the substrate wall to be of the form

$$w_o(\tau) = W_o\,\cos\Omega\tau, \qquad (W_o \ll 1) \tag{17}$$

Equation (15) then takes the form

$$\ddot{q}_j + \sum_i \left[\omega_{(j)}^2\delta_{ji} + \epsilon\,\cos\Omega\tau R_{ji}\right] q_i = -\epsilon\,\cos\Omega\tau\,V_j \qquad (j = 1,2,\dots) \tag{18}$$

where

$$\epsilon = 4\phi^2 W_o \ll 1 \tag{19}$$

$$V_j = \frac{2\phi}{C}\bar{P}_j\left(1 - \frac{\Omega^2}{4\phi^2}\right), \qquad (j = 1,2,\dots) \tag{20}$$

It may be noted that since $\tau = 1$ corresponds to the time required of a circumferential wave to traverse the delaminated segment of layer, Ω represents the number of cycles (divided by 2π) that the surface of the substrate goes through during this interval. Thus, Ω^2 will, in general, be a small quantity.

We note that when j corresponds to an anti-symmetric mode, V_j, and hence, the right side of Eq. (18) vanishes. Under such conditions, the resulting system falls into the class considered by Hsu,[10] and the regions of instability in the parameter space of the system can be determined directly. For the case when V_j does not vanish (i.e., j corresponds to a symmetric mode), the regions of instability may be found by a minor extension of the analysis presented in Ref. 10, which employs a combined perturbation and variation of parameters technique, to incorporate a nonvanishing and periodic right hand side of $\mathcal{O}(\epsilon)$. The regions of instability in $\epsilon - \Omega$ space are thereby found to be defined by the following inequalities:

When R_{ji} and R_{ij} are of the same sign:

$$\epsilon^2\Gamma_{ji}^2 > \left[\Omega - (\omega_j + \omega_i)\right]^2 \tag{21a}$$

When R_{ji} and R_{ij} are of opposite sign:

$$\epsilon^2\Gamma_{ji}^{*2} > \left[\Omega - (\omega_i - \omega_j)\right]^2 \qquad (i > j) \tag{21b}$$

when $i = j$

$$\epsilon^2\Gamma_{ii}^2 > (\Omega - 2\omega_i)^2 \tag{21c}$$

Conditions (21a-c) hold for all modes considered. In addition, the system becomes unstable for all $\epsilon \ll 1$ when the following equality is satisfied:

Symmetric modes only:

$$\Omega - \omega_j = 0 \tag{21d}$$

In the previous expressions,

$$\Gamma_{ji}^2 = R_{ji}R_{ij}/4\omega_j\omega_i \tag{22a}$$

$$\Gamma_{ji}^{*2} = -R_{ji}R_{ij}/4\omega_j\omega_i \tag{22b}$$

Solutions that correspond to any of the previous regions in the parameter space of the system are unstable. [It should be noted that as conditions (21a-c) correspond to a first-order retention of the perturbation parameter ϵ, it is likely that additional higher-order regions of instability exist. It was, however, pointed out by Hsu[10] in his analysis that such regions should be expected to be much less significant.]

Concluding Remarks

The problem of dynamic buckling of a delaminated segment of layer adhered to the surface of an oscillating cylindrical substrate has been considered and regions of instability in the parameter space of the system found.

As the deflections of the layer at the delamination edge coincide with those of the uniformly oscillating substrate boundary, no kinetic energy is released upon growth of the delamination and, thus, the energy release per unit angular extension of the disbond will be of the same general form as for the quasistatic case.[3-6] That is, it will be comprised of the sum of the relative bending and stretching energies at the delamination edge. For the solution considered herein, the energy release G takes the form

$$G\{\phi\} = \tfrac{1}{2}\sum_i q_i^2(\tau)\left[\bar{v}_i''^2(\phi) + \bar{P}_i^2/C\right] \tag{23}$$

When G reaches a critical value, growth of the disbond will occur. It is easily seen from this expression that this will happen when a single modal amplitude achieves a sufficient magnitude. Delamination growth can therefore be expected to occur when the parameters of the system lie in a region defined by Eqs. (21).

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